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PAN POLSKA AKADEMIA NAUK

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MEASUREMENT MODELS – THEORY AND PRACTICE OF UNCERTAINTY EVALUATION OF GEOMETRICAL DEVIATIONS MEASUREMENTS

Mirosław Wojtyła, Wojciech Płowucha

University of Bielsko-Biała, Laboratory of Metrology, Willowa 2, 43-309 Bielsko-Biała, Poland (✉ *mwojtyla@ubb.edu.pl*)

Abstract

The publication provides a critical analysis of fundamental documents concerning the determination of measurement uncertainty from the perspective of the machinery industry. The requirements contained in JCGM 104, JCGM 100, JCGM 101 documents were compared with important documents used in geometrical measurements, particularly with EA-4/02, ISO 14253-2, ISO/TS 15530-1, ISO 15530-3, ISO/TS 15530-4, and VDI/VDE 2617-11. Significant differences between the analysed documents, both terminological and interpretative, were highlighted. The analysis was performed in the sequence of stages of determining measurement uncertainty: formulation, propagation, and summarizing. Special attention was paid to the problem of defining the measurement model and the insufficient reference to the measurement model in the analysed documents. Attention was drawn to the wide range of characteristics measured in the machinery industry, such as linear and angular dimensions and form, orientation, position, and runout deviations, as well as the wide range of measurement equipment used, from simple instruments like callipers, micrometers, and mechanical dial gauges, to coordinate measuring machines and measurement systems. The current approach to the uncertainty of coordinate measurements, including the new possibility of modelling coordinate measurement, was discussed.

Keywords: uncertainty, law of propagation of uncertainty, methods of uncertainty propagation, sensitivity analysis, Monte Carlo method.

1. Introduction

A measurement result is generally expressed as a single measured quantity value and a measurement uncertainty. The measurement uncertainty is defined, among others in [1] as non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used. Measurement uncertainty may be presented as a standard deviation (called standard measurement uncertainty) or as a specified multiple of it (than is called expanded measurement uncertainty). Measurement uncertainty can be presented as a standard deviation (standard measurement uncertainty) or as a specified multiple of it (expanded measurement uncertainty).

Providing measurement uncertainty is mandatory for measurements and calibrations performed by accredited laboratories, as required by ISO 17025 [2]. Increasingly, the provision of measurement uncertainty is also expected for measurements performed in industry (especially automotive), particularly for measurements used to assess the conformity with the requirements. This is directly derived from the provisions of the ISO 14253-1 [3] and IATF 16949 [4].

Measurement in the mechanical engineering industry involves not only the manufactured parts but also the equipment, such as ring gauges [5], thread gauges [6], or taper gauges [7] (tapered parts are the gripping parts of cutting tools such as drills or reamers). The measured characteristics include linear and angular dimensions (see ISO 14405) [8, 9, 10] and geometrical deviations, such as form deviations (straightness, flatness, roundness, and cylindricity),

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orientation (parallelism, perpendicularity, and angularity), location (position, coaxiality, concentricity, and symmetry), and runout (see [11]).

Measuring equipment mainly consists of coordinate measuring machines and other coordinate measuring systems such as measuring arms or optical machines. Furthermore, simple measuring instruments like callipers, micrometres, mechanical dial gauges, and dial test indicators are still used. Specialized instruments are also utilized, such as roundness testers, gears measurement instruments, and surface roughness measurement devices.

The above information indicates the extensive knowledge required by individuals dealing with measurement uncertainty in mechanical engineering. Therefore, the industry anticipates publications containing examples that facilitate the development of procedures/instructions for determining the uncertainty of various types of measurements. A fundamental element of such an instruction should be the measurement model.

Measurement uncertainty should be a component of every measurement result; however, it is difficult to imagine that in industrial conditions, a full uncertainty analysis is conducted for every measurement.

In measurements of geometric quantities, the most crucial components of measurement uncertainty are often repeatability (especially in measurements performed under challenging conditions, such as difficult access to the measured surface or poor lighting), the thermal effects component, and the instrument-related component. The latter can be calculated based on the formula for *maximum permissible error* (MPE), provided by the instrument manufacturer and verified during periodic checks/calibrations. Therefore, a single analysis should suffice to obtain a valid uncertainty value over a more extended period, or at least to obtain a calculation scheme (*e.g.* in the form of a spreadsheet) within which only changing data, such as the temperature at which the measurement was performed, need to be provided to obtain the uncertainty value.

Guide to the expression of *uncertainty in measurement* (GUM) together with supplements organises largely the issues regarding measurements uncertainty determination. However, some ambiguity remains. This paper presents the evaluation of the current situation from the point of view of measurements conducted in mechanical engineering. In particular, requirements contained in JCGM 200 (ISO/IEC Guide 99, VIM) [1], JCGM 104 [12], JCGM 100 (GUM) [13], JCGM 101 [14], JCGM 102 [15] and JCGM GUM-6 [16] (theory) were compared with other documents applied in geometrical measurements, and in particular with EA-4/02M [17], ISO 14253-2 [18], ISO/TS 15530-1 [19], ISO 15530-3 [20], ISO/TS 15530-4 [21] and VDI/VDE 2617-11 [22] (practice). Attention was given, inter alia, to terminology, and, in particular, to terms such as: *GUM uncertainty framework* (GUF), law of propagation of uncertainty (LPU), type A and type B evaluations, three stages of uncertainty evaluation: formulation, propagation and summarising, explicit, implicit, extended, nested and multistage measurement models, propagation of distributions, three methods of propagation of uncertainty: analytical, GUF and Monte Carlo, application of central limit theorem, model linearization (Taylor expansion), correlation of input quantities, and others.

In the cited passages of the quoted documents symbols applied therein were kept and in most cases their meaning was not explained – it was assumed that these documents could be easily consulted. The article aims to evaluate the consistency and completeness of existing standards and documents on measurement uncertainty, especially in relation to geometrical measurements. Particular attention is paid to the availability of the measurement model, as it forms the basis for verifying the uncertainty budget.

The authors participate in ISO standardization work and in the development of a simple method for determining the uncertainty of coordinate measurements [23]. They also strive to disseminate more interesting examples of uncertainty analyses performed for the mechanical industry.

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2. Stages of measurement uncertainty evaluation

The main stages of uncertainty evaluation constitute formulation, propagation, and summarising [14, ch. 5.1], [12, ch. 5].

Formulation includes:

- defining the output quantity Y, the quantity intended to be measured (the measurand);
- determining the input quantity $X_1,\ldots, X_N$ upon which Y depends;
- developing a model relating Y and X;
- assigning (on the basis of available knowledge) *probability distribution functions* (PDFs) to particular X_i; assigning a joint PDFs to those X_i that are not independent.

Propagation consists in determination of the output quantity distribution on the basis of the model and the input quantities PDFs.

Summarising: using the PDF for Y to obtain (depending on the need) calculation (from PDF of the quantity Y) all or some of the below:

- the expectation of Y, taken as an estimate y of the quantity,
- the standard deviation of Y, taken as the standard uncertainty $u(y)$ associated with y,
- a coverage interval containing Y with a specified probability (the coverage probability).

In measurements of geometric quantities, the output quantity Y is the measured characteristic of the object, such as a dimension or geometric deviation. Most often, direct measurement involves one input quantity X_1, which is the one being measured (in the measurement of the shaft diameter with a micrometer, both the output and input quantities are the shaft diameter, meaning $Y = X_1$). However, if we want to take into account the fact that the measurement result is affected by elastic deformation caused by the micrometer's measuring force, a second input quantity X_2 appears – the correction related to the deformation. Since the correction should be added to the raw (uncorrected) measurement result, the relationship between Y and X_1 and X_2 (the measurement model) will take the form $Y = X_1 + X_2$.

The diameter of the shaft can be measured using coordinate measuring technique, such as with a *coordinate measuring machine* (CMM). From the user's perspective, this measurement can be treated similarly to a micrometer measurement (after completing the measurement program, the diameter value is obtained). However, upon closer inspection (analysing the part program), it can be observed that the measurement is, in fact, an indirect measurement: the coordinates of a certain number of points on the shaft's surface are "measured", and from the obtained values, the diameter is calculated. This means that the measurement involves a certain number of input quantities $X_1, ..., X_n$, and the measurement model takes the form of an unknown (non-linear) function: $Y = f(X_1, ..., X_n)$.

Let's imagine another case. The aim of the measurement is the diameter of a hole in a flat object. We have a measurement microscope that is not equipped with a computer. However, we decide to perform the measurement using the coordinate technique by "measuring" the coordinates of 3 points: x_A, y_A, x_B, y_B, x_C, and y_C (Fig. 1).

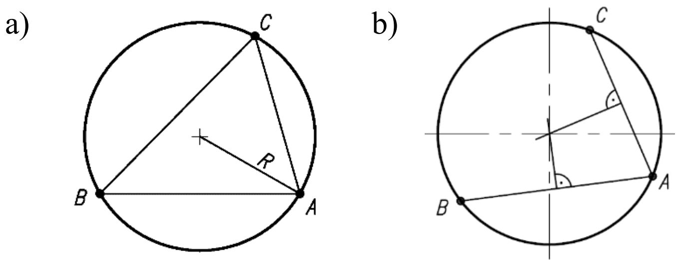

Fig. 1. Illustration of different models for coordinate measurement of circle radius: a) radius calculated as the radius of a circle circumscribed on a triangle, b) centre of the circle determined as the intersection point of the perpendicular bisectors of the triangle's sides.

There are several possibilities for calculating the radius of a circle, which is equivalent to saying that there are several possible measurement models. Two of them are presented. The first is to use the well-known mathematical formula for the radius of a circle circumscribed around a triangle using Heron's formula for the area of the triangle. The input quantities here are the lengths of the sides a, b, c calculated from the measured differences in the coordinates of the pairs of points *B-C, A-C*, and *B-A*:

$$\begin{gathered} R = \frac{abc}{2\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}} \\ a = \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2} \\ b = \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2} \\ c = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \end{gathered} \tag{1}$$

The second possibility is to determine the centre of the circle as the intersection point of its perpendicular bisectors.

$$\begin{gathered} R = \sqrt{[0.5(x_B - x_A) + t(y_B - y_A)]^2 + [0.5(y_B - y_A) - t(x_B - x_A)]^2} \\ t = \frac{1}{2}\frac{(x_C - x_B)(x_C - x_A) + (y_C - y_B)(y_C - y_A)}{(y_B - y_A)(x_C - x_A) - (y_C - y_A)(x_B - x_A)} \end{gathered} \tag{2}$$

It can be assumed that both measurement models involve six input quantities: x_A, y_A, x_B, y_B, x_C, and y_C (coordinates of points *A*, *B*, and *C*). However, it will turn out that instead of coordinates, it is better to assume certain differences of coordinates as input quantities (also six). In the first case, these are: x_B-x_C, y_B-y_C, x_A-x_C, y_A-y_C, x_B-x_A, and y_B-y_A, in the second: x_B-x_A, y_B-y_A, x_C-x_B, y_C-y_B, x_C-x_A, and y_C-y_A. This is due to the fact that for length measuring devices *MPE* applies to the length value regardless of the location of this length in the measuring range.

Above, different example measurement models of the same output quantity: the diameter or radius of the circle, are presented. It is already evident that an important and at the same time difficult element of the measurement uncertainty evaluation procedure is the development of the measurement model.

3. General notation of the measurement model

A measurement model, as an essential element enabling the determination of measurement uncertainty, is discussed in many documents. In the GUM, the term "measurement model" does not appear; instead, the term "mathematical model" [13, ch. 3.1.6, ch. 3.4.1] is used as a shorthand for "mathematical measurement model" [13, ch. 3.4.2]. The term "measurement model" (or its shortened form "model") is commonly used in JCGM 104 [14, ch. 3.10], referring to its definition from VIM [1, ch. 2.48]: "*mathematical relation among all quantities known to be involved in a measurement.*" It is worth mentioning that VIM contains many other terms related to measurement uncertainty.

The general formula for the measurement model has the following form [12, ch. 3.16] $h(Y, X_1, \ldots, X_N) = 0$, and it can be most frequently presented in the form of a measurement function [12, ch. 3.15] $Y = f(X_1, \ldots, X_N)$. In JCGM 101 [14] the same measurement models have vector notation [14, ch. 4.1] $h(Y, \boldsymbol{X}) = 0$, $Y = f(\boldsymbol{X})$.

In JCGM 102 [15] a possibility of the output quantity being a vector quantity is considered, and then we talk about a multivariate measurement model, which has a general form [15, ch. 3.8] $h(\boldsymbol{Y}, \boldsymbol{X}) = 0$ and in case where it can be presented as a measurement function $\boldsymbol{Y} = f(\boldsymbol{X})$.

The basic application of this measurement model relates to complex quantities. In geometric measurements such an approach can be used as regards geometrical elements which are defined by a particular number of parameters (vector). For example, to define a 2D circle 3 parameters are needed: two coordinates of the centre and a radius.

The simplest form of the measurement model (function) is $Y = X$ where the output quantity is simply equal to the reading of a measuring instrument. Frequently, when it is possible to indicate additive sources of error, *e.g.* when it is possible (although not necessarily applied) to correct systematic errors, the measurement function takes the form $Y = X + \delta_1 + \cdots + \delta_n$. Then we talk about the extended measurement model [16, ch. 10]. In general, when the output quantity is a sum of input quantities $Y = X_1 + \cdots + X_N$ we talk about the additive model [14, ch. 9.2]. In practice, linear measurement functions act frequently as models, that is, functions of the form [12, ch. 4.14] $Y = c_1 X_1 + \cdots + c_N X_N$.

Moreover, nonlinear models can be linearized [16, Annex F] by, for example, expanding the function into a Taylor series and rejecting higher-order terms. In some cases, to obtain proper accuracy it is necessary to leave some higher-order terms. Such a situation occurs, among others, in threads measurements.

4. Measurement model in other important documents

From the area of geometrical measurements, we find in JCGM 100 a measurement model of the gauge block length taking into account the effect of the thermal expansion phenomenon [13, ch. H.1.2] in the form

$$l = \frac{l_s(1+\alpha_s\theta_s)+d}{1+\alpha\theta} = l_s + d + l_s(\alpha_s\theta_s - \alpha\theta) + \cdots \tag{3}$$

where: d is the difference in the lengths of the gauge block being calibrated and the standard, l is the measurand, that is, the length at 20 °C of the gauge block being calibrated, l_S is the length of the standard at 20 °C as given in its calibration certificate, α and α_S are the coefficients of thermal expansion, respectively, of the gauge being calibrated and the standard, θ and θ_S are the deviations in temperature from the 20 °C reference temperature, respectively, of the gauge block and the standard.

The first part of the above formula is an exact formula, and the second part is its linearization (a fragment of the Taylor series expansion). The formula clearly indicates the significant importance of temperature differences and differences in expansion coefficients (the two error components from thermal effects are more visible in it).
The other form of the model is

$$l = f(l_s, d, \alpha_s, \theta, \delta\alpha, \delta\theta) = l_s + d - l_s(\delta\alpha \cdot \vartheta - \alpha_s \cdot \delta\theta) \tag{4}$$

where $\delta\alpha = \alpha - \alpha_s$ i $\delta\theta = \theta - \theta_s$.

Gauge blocks are among the most used length standards. Their measurements (calibration) are of interest to calibration laboratories and are the subject of numerous publications [*e.g.*, 24].

In the document EA-4/02M [17] the measurement model is most often referred to as “model function” or ”relation”. We find there 3 examples of models from the area of gauges and geometrical measurement instruments calibration:

- calibration of a gauge block of the nominal length of 50 mm [17, ch. S4]:

$$l_X = l_S + \delta l_D + \delta l + \delta l_C - L(\bar{\alpha} \cdot \delta t + \delta\alpha \cdot \Delta\bar{t}) - \delta l_V \tag{5}$$

where: l_X is the length of the unknown gauge block, l_S is the length of the reference gauge block at the reference temperature t_0 = 20 °C according to its calibration certificate, δl_D is the change of the length of the reference gauge block since its last calibration due to drift, δl is the observed difference in length between the unknown and the reference gauge block, δl_C is the correction for non-linearity and offset of the comparator, L is the nominal length of the gauge blocks considered; $\bar{\alpha}$ is the average of the thermal expansion coefficients of the unknown and reference gauge blocks, δt is the temperature difference between the unknown and reference gauge blocks, $\delta\alpha$ is the difference in the thermal expansion coefficients between the unknown and the reference gauge blocks, $\Delta\bar{t}$ is the deviation of the average temperature of the unknown and the reference gauge blocks from the reference temperature, δl_V is the correction for non-central contacting of the measuring faces of the unknown gauge block;

- calibration of vernier calliper [17, ch. S10]:

$$E_X = l_{iX} - l_S + L_S \cdot \bar{\alpha} \cdot \Delta t + \delta l_{iX} + \delta l_M \tag{6}$$

where: E_X is the error of indication l_{iX} is the indication of the calliper, l_S is the length of the actual gauge block, L_S is the nominal length of the actual gauge block, $\bar{\alpha}$ is the average thermal expansion coefficient of the calliper and the gauge block, Δt is the difference in temperature between the calliper and the gauge block, δl_{iX} is the correction due to the finite resolution of the calliper, δl_M is the correction due to mechanical effects, such as applied measurement force, Abbe errors, flatness and parallelism errors of the measurement faces;

- calibration of a ring gauge of the nominal diameter of 90 mm [17, ch. S13]:

$$d_X = d_S + \Delta l + \delta l_i + \delta l_T + \delta l_P + \delta l_E + \delta l_A \tag{7}$$

where: d_X is the diameter of the ring, d_S is the diameter of the reference setting ring at the reference temperature, Δl is the observed difference in displacement of the measuring spindle when the contact tips touch the inner surface of the rings at two diametrically apart points, δl_i is the correction for the errors of indication of the comparator, δl_T is the correction due to the temperature effects of the ring to be calibrated, the reference setting ring and the comparator line scale, δl_P is the correction due to coaxial misalignment of the probes with respect to the measuring line, δl_E is the correction due to the difference in elastic deformations of the ring to be calibrated and the reference setting ring, δl_A is the correction due to the difference of the Abbe errors of the comparator when the diameters of the ring to be calibrated and the reference setting ring are measured.

Let us look at those models in more detail. Significant inconsistency is noticeable in treating the temperature error. In the first example there are two components (types) of this error: the first one ($L \cdot \bar{\alpha} \cdot \delta t$) is derived from the difference in temperature of the workpiece and the instrument (δt), and the other one ($L \cdot \delta\alpha \cdot \Delta\bar{t}$) from the difference of expansion coefficients ($\delta\alpha$) and deviation from the reference temperature ($\Delta\bar{t}$). In the second example only the first of these two components is present ($L_S \cdot \bar{\alpha} \cdot \Delta t$), that is, the difference in temperature of the gauge block and the vernier calliper (Δt) (one can guess that the other component was omitted as insignificant). The third example differs from the previous ones in that two rings and a comparator participate in the measurement. In the model the temperature error is present jointly as δl_T (”*correction due to the temperature effects of the ring to be calibrated, the reference setting ring and the comparator line scale*”), four components of this error were provided, and a separate analysis was used to determine the uncertainty component related to this error (”uncertainty *sub-budget*”).

The term ”measurement model” is not used in ISO 14253-2 [18]. Instead, the term ”model of uncertainty estimation” is commonly used, within which a distinction is made between the black box method and the transparent box method. In the black box method of uncertainty estimation the result of the measurement is the reading corrected by an eventually known correction (that is, the measurement model has the form):

$$Y = X + C. \tag{8}$$

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In the transparent box method of uncertainty estimation, the value of the measurand is modelled as a function of several measured values X_i, which themselves could be functions (transparent box models) or black box models, or both: (that is, the measurement model has the form):

$$Y = G\left(X_1, X_2, \cdots, X_{p+r}\right). \tag{9}$$

As a side note: $p+r$ index was applied to emphasise that among input quantities there is p uncorrelated quantities and r correlated quantities.

In the standard there are the following four examples of uncertainty estimation for:

- calibration of a setting ring,
- measurement of local diameter with an external micrometer,
- calibration of error of indication of an external micrometer,
- measurement of roundness.

The second example mentioned, with some completions, can serve as a basis for developing a guideline for determining the uncertainty of direct measurements performed in industry.

The following formula for the measurement uncertainty was provided:

$$u_c = \sqrt{u_{ML}^2 + u_{MF}^2 + u_{MF}^2 + u_{MP}^2 + u_{RR}^2 + u_{NP}^2 + u_{TD}^2 + u_{TA}^2 + u_{WE}^2}. \tag{10}$$

The individual uncertainty components are: u_{ML} – micrometer –error of indication, u_{MF} – micrometer – flatness of measuring anvils, u_{MP} – micrometer – parallelism of measuring anvils, u_{RR} – resolution u_{RA} or repeatability u_{RE} (the largest of the two), u_{NP} – variation of zero point between the operators, u_{TD} – temperature difference, u_{TA} – difference in thermal expansion coefficients and the deviations in temperature from the 20 °C reference temperature, u_{WE} – workpiece form error.

The given formula for measurement uncertainty emphasizes the fact that individual components of uncertainty are added in a quadratic manner. This indicates the presence of an extended measurement model, which can be reconstructed in the following manner:

$$l = l_M + \delta_{MF} + \delta_{MF} + \delta l_{MP} + \delta l_{RR} + \delta_{NP} + \delta l_{TD} + \delta l_{TA} + \delta l_{WE}. \tag{11}$$

It is worth paying attention to the fact that uncertainty components are grouped consecutively according to their 4 sources: instrument (4 components), human (2 components), environment (2 components) and measured workpiece (1 component). Relevant uncertainty components come consecutively from error of indication, flatness of measuring anvils (two identical), parallelism of measuring anvils, repeatability or resolution, variation of zero point, temperature difference between micrometer and workpieces, deviation from standard reference temperature and error of form deviation of the workpiece. Two components (those the source of which is a metrologist) were determined with the A type method, the remaining ones with the B type method.

In case of the B type method the highest values that could be assumed by particular errors were determined.

The standard uncertainty type A is determined experimentally. For the type B method, the highest values that particular errors can assume are determined (denoted by a), and the appropriate probability distribution for this random variable is identified. Each specific probability distribution is associated with a coefficient b which allows the conversion of value a to the value of standard uncertainty u [18, ch. 8.3.2]

$$u = a \cdot b. \tag{12}$$

Usually, one of the following distributions is chosen (the coefficient b value is given in parentheses): normal (usually 0.5), uniform (also called rectangular or even, 0.58), triangular (0.41), U-shaped (typically arcsine distribution, 0.71). The provided (approximate) values for the coefficient *b* result from the relationship between the parameter *a* value and the variance of the distribution.

For the components derived from the instrument (micrometer) information on *MPE* (maximum permissible error of indication), flatness and parallelism of anvils provided by the manufacturer was used. Two components deriving from the environment (due to thermal influences, to be more precise) were considered. For the first one (temperature difference between micrometer and workpieces) the extreme value a_{TD} was calculated on the basis of earlier observations that this difference does not exceed 10 °C. For the other one (following from the differences between linear coefficients of thermal expansion and the fact that measurements are not conducted at the temperature of 20 °C) the extreme value a_{TA} was calculated based on earlier observations that maximum deviation from standard reference temperature is 15 °C and on the assumption that linear coefficients of thermal expansion differ by maximum 10%. The extreme value for the component deriving from the workpiece was calculated based on the observation that cylindricity does not exceed 1.5 μm.

The standard contains a minor error instead of $a_{TA} = 0.1 \times \Delta T_{20} \times \alpha \times D$ there should be $a_{TA} = \Delta T_{20} \times \Delta\alpha \times D$ (only in the example $\Delta\alpha = 0.1\alpha$)

Attention needs to be paid to the fact that in the current standard on the requirements for micrometers [25] MPE_{MF} and MPE_{MP} are no longer present, which results in the need for modification of the example in future revision of the standard.

The technical specification ISO/TS 15530-1 [19] (*Technique for determining the uncertainty of measurement. Part 1: Overview and metrological characteristics*) mentions complying with GUM and ISO 14253-2. With reference to the specificity of the coordinate measuring technique attention was drawn to ”three general uncertainty categories”. The first one is instrumentation factors. These factors are typically the responsibility of the CMM manufacturer and are controlled by establishing permissible limits, *e.g.* temperature ranges, under which the CMM may be used. Some or all of these error sources may be assessed during acceptance or reverification testing of the CMM. The second category is measurement plan factors which involve how the CMM user decides to execute the measurement. This includes the workpiece location and orientation, the probes and styli selected for the measurement, and the particular measurement point sampling strategy. In this category attention is paid to the fact that the quantity being measured shall be unambiguously specified (it relates, among others, to matching criteria: a least-squares, minimum-circumscribed, maximum-inscribed or minimum-zone). The third category is extrinsic factors such as non-ideal workpiece geometry (surface roughness, form errors, finite stiffness and thermal distortions), contamination, workpiece fixturing problems and variations among operators.

Unfortunately, despite using the term ”task specific uncertainty”, no explicit attention is paid to the large diversity of geometrical characteristics of the measured workpieces (dimensions, angles and deviations of form, orientation, location and runout) [11].

The document distinguishes 3 techniques of coordinate measurements uncertainty determination. The first one, ”*sensitivity analysis*” refers clearly to GUF. The second one, however, ”*use of calibrated workpieces or measurement standards*” is treated as a technique not present in GUM, whereas it is fully compliant with GUF. The third technique, ”*using simulation*” refers to uncertainty propagation with the Monte Carlo method present in GUM. As a side note, it is worth noting that the Monte Carlo method can also be used to determine uncertainty components. The simulation of the measurement of the circle diameter of the workpiece with the three-lobed form error with the application of sampling in 6 uniformly distributed points, described in [21, Annex F], can serve as an example. The obtained distribution of errors is not a normal distribution and does not even contain the true value. More information can be found in [26].

In ISO 15530-3 [20] (*Technique for determining the uncertainty of measurement. Part 3: Use of calibrated workpieces or measurement standards*) the term ”measurement model” is not used. From the contents, and in particular from the provided formula for the measurement uncertainty calculation:

$$U = k \cdot \sqrt{u_{cal}^2 + u_p^2 + u_b^2 + u_w^2} \tag{13}$$

where: u_{cal} is the standard uncertainty associated with the uncertainty of the calibration of the calibrated workpiece stated in the calibration certificate, u_{p} is the standard uncertainty associated with the measurement procedure as assessed below, u_{b} is the standard uncertainty associated with the systematic error of the measurement process evaluated using the calibrated workpiece, u_{w} is the standard uncertainty associated with material and manufacturing variations (due to the variation of expansion coefficient, form errors, roughness, elasticity and plasticity).

It can however be inferred that GUF was applied and that we are dealing with an extended model in the form:

$$Y = X + \delta_{cal} + b + \delta_w \tag{14}$$

in which the output quantity is encumbered with 3 errors (δ_{cal}, b and δ_{w}). From the provisions of the standard it also follows that b shall be treated as a systematic (corrected) error with uncertainty u_{b}, whereas δ_{cal} and δ_{w} shall not be corrected. Standard deviation calculated from the results of 20 repetitions of the measurement is treated as a component of the measurement uncertainty u_{p} determined with the type A evaluation (the standard provides a long list of factors having impact on this uncertainty component). As a side note, it is worth considering whether any other number of the measurement repetitions and application of t distribution should not be provided for in the standard.

The component u_{cal} is determined with the B method. The standard states that the component u_{w} can be determined with the type A or type B evaluation. It also states that the component u_{w} shall be calculated as a geometric sum of two components u_{wt} and u_{wp}, but there is no convincing argument in favour of the wisdom of their application. The overall conclusion is that there are significant inconsistencies in comparison with GUM.

From the title of the technical specification ISO/TS 15530-4 [21] (*Evaluating task-specific measurement uncertainty using simulation*) it can be incorrectly inferred that the document contains recommendations as regards design of a relevant simulation model, while in reality we only find requirements to be met by simulation software as well as by any other software used for determination of coordinate measurements uncertainty, generally referred to as ”*uncertainty evaluating software* (*UES*)”.

The only formula which is related to the measurement model is:

$$u = \sqrt{u_{sim}^2 + \sum u_i^2} \tag{15}$$

saying that an uncertainty component determined with the simulation technique should be complemented by components obtained otherwise.

The title of the document VDI/VDE 2617:11 [22] (Determination of the uncertainty of measurement for coordinate measuring machines using uncertainty budgets) refers to GUF through the words “uncertainty budget”. The term “mathematical model” [22, ch. 3.2.2] and other terms present in GUM, such as, for example, type A or type B evaluation, standard uncertainties or sensitivity coefficients can be found in the document. The document indeed describes in detail a method of coordinate measurements uncertainty estimation. Two measurement models are also provided. The first one refers to the uncertainty of measurement of an bore diameter and has the form (symbols according to [22]):

$$D = (D_W - \Delta D_T + \Delta D_C) - \Delta L_{CMM} - \Delta L_t \tag{16}$$

where: D_W is the diameter of the regression feature on the workpiece, ΔD_T is the error of stylus diameter during stylus qualification, ΔD_C is the error of the calibrated diameter of the reference standard, ΔL_{CMM} is the geometrical error of the CMM, ΔL_t is the length error due to the error of the expansion coefficient of the scale.

The other model refers to the measurement of the distance between a plane and a cylinder axis (symbols according to [22]):

$$L = \left(X_E - W_E \frac{L_{SE}}{L_{ME}} + \Delta X_{TE} - \Delta R_{TE} + \frac{\Delta D_C}{2}\right) +$$

$$-\left(X_B - W_B \frac{L_{SB}}{L_{MB}} + \Delta X_{TB} - \Delta R_{TB} + \frac{\Delta D_C}{2}\right) - (-\Delta X_{TR} - \Delta L_{CMM} - \Delta L_t) \tag{17}$$

where: X_E is the coordinate of the toleranced feature at the centre of gravity, W_E is the angle error of the toleranced feature for the measured length, L_{ME}, ΔX_{TE} is the error of the centre coordinate of the stylus for the toleranced feature during qualification, ΔR_{TE} is the error of the radius of the stylus for the toleranced feature, X_B is the coordinate of the reference feature at the centre of gravity, WB is the angle error of the reference feature for the measured length, L_{MB}, ΔX_{TB} is the error of the centre coordinate of the stylus for the reference feature during qualification, ΔR_{TB} is the error of the radius of the stylus for the reference feature, ΔD_C is the error of the calibrated diameter of the reference standard, ΔX_{TR} is the error of the centre coordinates of the styli due to rotation errors, ΔL_{CMM} is the geometrical error of the CMM.

For the mentioned models example uncertainty budgets are also provided. The problem is that this method has not been widely recognised due to complex theoretical principles.

5. Multistage measurement models

Here it may be worth referring to the term ”multistage model” occurring in JCGM 104 [12]. In the multistage model output quantities from previous stages become input quantities to subsequent stages. In the first example two input quantities are measured with the use of various measuring systems of the same instrument and it cannot be ruled out that each of them is measured with a different uncertainty. In the second example two input quantities are measured with various measuring instruments. In the third example we are dealing with two types of quantities: the output quantity is an angle and the input quantities are lengths. In the fourth example, in the first step (stage) measurement uncertainties for 3 lengths *a*, *b* and *c* need to be determined. It is worth mentioning here that information concerning the accuracy of these lengths’ measurement is contained in the formula for maximum permissible error for length measurement $E_{\mathrm{L,MPE}}$ provided by the manufacturer [27, ch. 3.6]. Additional measurement models may be needed to evaluate the measurement uncertainty of the input quantities. In the last two examples some functions of the directly measured quantities were assumed as input quantities. In all cases it is possible (or necessary) to use two-stage models. One of the two needed lower-stage models for the Fig. 1b measurement may be a model similar to the model of measuring the shaft diameter with a micrometer from ISO 14253-2 [18].

Two-stage models can also be all previously provided models in which ”corrections” were used, and there was no information whether they are actually applied – if they are, then the first-stage model shall be needed to evaluate their uncertainty.

It should be noted that the first two models as referred to in EA-4/02M [17, formulae 8 and 9] would be more elegant if the components related to the temperature error were relocated to the lower stage, that is, if the highest stage models assumed the form (see (6)):

$$l_X = l_S + \delta l_D + \delta l + \delta l_C + \delta l_T - \delta l_V, \tag{18}$$

$$E_X = l_{iX} - l_S + \delta l_T + \delta l_{iX} + \delta l_M. \tag{19}$$

6. Measurement models in the coordinate measuring technique

Currently, the basic technique used for geometrical measurements commonly applied in the industry is the coordinate measuring technique. This name means measurements with such measuring instruments as coordinate measuring machines, coordinate measuring arms, *Computed Tomography* (CT) scanners, laser-trackers, and others [28].

In ISO/TS 15530-4 [21] there is a statement: "... in the case of a CMM, the formulation of a classical uncertainty budget is impractical for the majority of the measurement tasks due to the complexity of the measuring process". This probably concerns less an uncertainty budget, since in an extreme case it can contain only one or two components including jointly a significant number of influencing factors, than the fact that it is impossible to demonstrate how individual factors, known in particular to CMM manufactures, influence measurement uncertainty. This concerns in particular such factors as individual geometrical errors (and usually 21 are listed) or a measuring head errors. Interest in the influence of these errors is connected with previous research aiming at their mathematical correction mastered by manufacturers.

A little bit further there is a statement that one of the alternative methods (for an uncertainty budget?) is using UES available on the market "... *based on a computer-aided mathematical model of the measuring process. In this model, the measuring process is represented from the measurand to the measurement result, taking important influence quantities into account.*". The best known measurement model is the measurement model of single point coordinates. This model follows from CMM design (geometry, kinematics) and assumes the existence (in the simplest case) of 21 geometrical errors, among which only 3 (perpendicularity errors) are single random variables, and the remaining 18 are functions of CMM's three measuring systems readings. In the most general version the model has the form (symbols according to [29]):

$$\boldsymbol{x}^* = \boldsymbol{x}_x(x, \boldsymbol{b}) + \boldsymbol{R}_x(x, \boldsymbol{b})\{\boldsymbol{x}_y(y, \boldsymbol{b}) + \boldsymbol{R}_y(y, \boldsymbol{b})[\boldsymbol{x}_z(z, \boldsymbol{b}) + \boldsymbol{R}_z(z, \boldsymbol{b})]\} + \boldsymbol{\varepsilon} \tag{20}$$

where $\boldsymbol{x}_\mathrm{x}$, $\boldsymbol{x}_\mathrm{y}$ and $\boldsymbol{x}_\mathrm{z}$ mean matrices of translation errors, and $\boldsymbol{R}_\mathrm{x}$, $\boldsymbol{R}_\mathrm{y}$ and $\boldsymbol{R}_\mathrm{z}$ matrices of rotation errors (nesting results from CMM design; it was assumed that the unit connected with the axis *z* moves together with the unit connected with the axis *x* and both of them together along the axis *y*).

In another publication [30], for a specific CMM solution (Fig. 2) an analogous model (this time instead of the vector of the point coordinates encumbered with a measurement error the output quantity is vector $\boldsymbol{E}$, that is, the point measurement error) is presented in a simplified form:

$$\boldsymbol{E} = \boldsymbol{P} + \boldsymbol{A} \cdot \boldsymbol{X} + \boldsymbol{A}_P \cdot \boldsymbol{X}_P \tag{21}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & -ywz - xrz & zwx + xry + yry \\ 0 & 0 & -zwy - xrx - yrx \\ 0 & xrx & 0 \end{bmatrix}, \tag{22}$$

$$\boldsymbol{A}_P = \begin{bmatrix} 0 & -xrz - yrz - zrz & xry + yry + zry \\ xrz + yrz + zrz & 0 & -xrx - yrc - zrx \\ -xry - yry - zry & xrx + yrx + zrx & 0 \end{bmatrix}, \tag{23}$$

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \boldsymbol{X}_P = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}, \boldsymbol{P} = \begin{bmatrix} xtx + ytx + ztx \\ yty + xty + zty \\ ztz + ztx + zty \end{bmatrix}. \tag{24}$$

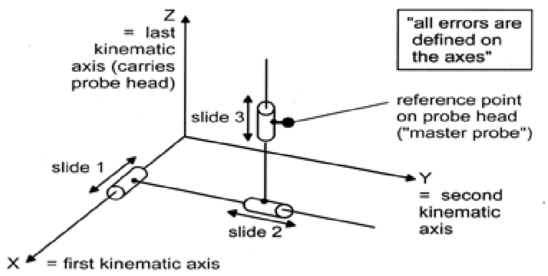

Fig. 2. CMM kinematic diagram (source: [30], original description).

The above models constitute a basis (they are first stage models) for evaluation of individual characteristics measurement uncertainty (dimensions, geometrical deviations) measured with the coordinate technique. In the second stage geometrical features (*e.g.* planes, cylinders) are matched to the gathered points, that is, parameters of these features (a plane – 6 parameters: a point and a standard unit vector, a cylinder – 7 parameters: a point, a standard unit vector of an axis and a diameter or a radius) are the output quantities. Assuming that the output quantities from the first stage model is *n* coordinates of points x_i, y_i, z_i, a general second stage model for a plane has the form:

$$\begin{bmatrix} x_P \\ y_P \\ x_P \\ u \\ v \\ w \end{bmatrix} = f(x_i, y_i, z_i), i = 1 \cdots n. \tag{21}$$

In case of dimensions and deviations of form at the second, and in other cases at the third stage relevant characteristics are calculated. Here the output quantity is a scalar (value of the dimension or the geometrical deviation).

For example, the value of the flatness deviation *flt* is calculated from the same input quantities as the plane parameters:

$$flt = f(x_i, y_i, z_i), i = 1, \cdots, n \tag{26}$$

but one of a few options to calculate the value of the parallelism of two planes *prl* is to use the parameters *u*, *v*, *w* of both planes and also the size of the one which is a tolerated element (information contained in the coordinates of the points can be used to approximately evaluate the size *r* of the feature):

$$prl = f(u_1, v_1, w_1, u_2, v_2, w_2, r). \tag{27}$$

Information on the geometrical errors of CMM (data for the first stage model) can be obtained on the basis of a rather labour-intensive experiment. Only the producer of CMM software have full knowledge concerning the applied second and third stage models. Knowledge of these models shall not be necessary if the Monte Carlo method is used at the propagation stage and there is a possibility to use CMM software.

In several European metrological institutes, simulation software developed at PTB and known as VCMM is used. Information on the probability distributions of errors occurring in the machine model is obtained based on several days of CMM research under good environmental conditions of the given laboratory. This information must be periodically updated, which limits the area of application to calibration laboratories. Software for use in industry must allow the simulation of measurements under variable environmental conditions [*e.g.*, 31].

In ISO/TS 15530-1 [19, ch. 6.2], the following statement is outdated: "*Since CMMs are complex measuring instruments, directly implementing this technique may only be possible for* *a limited number of measuring tasks*". That statement is now outdated. In many publications, and in particular in [32, 33], the opposite has been shown. With appropriate assumptions, the measurement model (of a not too complex form) can include the essence of the coordinate measurements. The sensitivity coefficients present in the uncertainty budgets obtained on the basis of these models allow for an unambiguous indication of the weight of individual components.

The mentioned method/technique (called in [32, 33] "sensitivity analysis") allows to determine the uncertainty for all geometrical characteristics, both for linear and angular dimensions, as well as for all geometrical deviations (form, orientation, location and runout) [11]. These models are based on known formulae [34, Table B.7] (symbols according to [34]):

- point – straight line distance

$$d(PT_1, SL_2) = |(\boldsymbol{A}_2 - \boldsymbol{PT}_1) \times \boldsymbol{u}_2| \tag{22}$$

- point – plane distance

$$d(PT_1, PL_2) = |(\boldsymbol{A}_2 - \boldsymbol{PT}_1) \cdot \boldsymbol{u}_2| \tag{23}$$

- point – point distance

$$d(PT_1, PT_2) = |\boldsymbol{PT}_1 - \boldsymbol{PT}_2| \tag{24}$$

- straight line – straight line distance

$$d(SL_1, SL_2) = \left|(\boldsymbol{A}_2 - \boldsymbol{A}_1) \cdot \frac{(\boldsymbol{u}_1 \times \boldsymbol{u}_1)}{|\boldsymbol{u}_1 \times \boldsymbol{u}_1|}\right| \tag{25}$$

As an example a measurement model of the position of the point (centre of the sphere) relative the secondary datum in the form of two perpendicular planes was provided (Fig. 3).

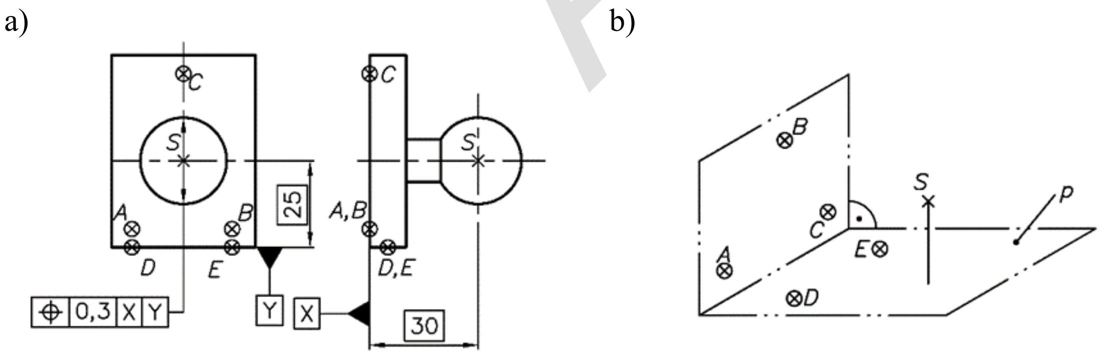

Fig. 3. Measurement model of the position of the point from the secondary datum in the form of two perpendicular planes: a) a technical drawing, b) characteristic points for the measurement model.

To define a measurement model a mathematically minimal number of points necessary to calculate the distance of the point S (centre of the sphere) from the plane *p* constituting the secondary datum is selected. The position is equal to the doubled value of the distance difference (observed and theoretically exact dimension, in the example equal to 25 mm) and it can be expressed as a function of differences of coordinates of 4 pairs of points (AB, AC, DE and DS), that is a function of 12 input quantities:

$$l(\boldsymbol{AB}, \boldsymbol{AC}, \boldsymbol{DE}, \boldsymbol{DS}) = 2\left(\boldsymbol{DS} \cdot \frac{(\boldsymbol{AB} \times \boldsymbol{AC}) \times \boldsymbol{DE}}{|(\boldsymbol{AB} \times \boldsymbol{AC}) \times \boldsymbol{DE}|} - 25\right). \tag{32}$$

So far, over 20 measurement models have been developed and programmed (Python, off-line version), enabling the evaluation of measurement uncertainty for most geometric measurements. Currently, research is underway on the online version.

7. Assignment of probability distributions

From the provisions in JCGM 104 [12] and JCGM 101 [14] it follows that to all input quantities probability distributions which are relevant for them need to be assigned. It is not an accurate wording, since in some cases knowledge of a standard deviation is enough. It is very often assumed that input quantities have a normal distribution, less frequently *t*-distribution. It is often assumed that input quantities have a symmetrical distribution (in case of errors distribution – symmetrical in relation to zero). There are however cases, where it is known, for example, that a given quantity or a measurement error of this quantity assume only non-negative values. Examples of such quantities in the mechanical engineering are all geometrical deviations (deviations of form, orientation, location and runout). Identification of the form of probability distribution and/or estimation of distribution parameters of any random variable can be performed based on research/statistical analyses or based on other information. Unfortunately, documents on uncertainty practically do not address this subject. GUM allows that "*using available knowledge*" [13, ch. 3.3.5], "*for insight based on experience and general knowledge*" [13, ch. 4.3.2], "*an a priori distribution*" [13, ch. 4.1.6] or "*available information*" [14, Table 1] to assume that a given input quantity has a specific distribution (*e.g.* uniform, triangular, normal, arcsine (U)). It also permits approximate assumption of extreme values which a given random variable can assume [18, Table B.1]). In the document JCGM 101 [14] the following probability distributions (except for those mentioned previously) are described in some detail (random numbers generators included): trapezoidal, curvilinear trapezoidal, exponential, gamma and multivariate normal (Gaussian) distribution. Sometimes a histogram, also referred to as frequency distribution, is enough to describe a random variable [14]. The problem of assigning probability distributions to input quantities is related to the terms "*type A uncertainty*" and "*type B uncertainty*": "*Type A evaluations of standard uncertainty components are founded on frequency distributions while Type B evaluations are founded on a priori distributions.*" used in GUM. It is worth mentioning here as a side note that type A and type B evaluations are methods of determining uncertainty components, and not generally understood methods of uncertainty determination. Both types of evaluation are based on probability distributions, and the uncertainty components resulting from either type are quantified by variances or standard deviations [13, ch. 3.3.4].

8. Propagation of distributions

Propagation of distribution is the method used to determine the probability distribution for an output quantity from the probability distributions assigned to the input quantities on which the output quantity depends [14, ch. 3.17]. In a general case propagation can be implemented with three methods: analytically, using the central limit theorem (in accordance with GUF) or with the Monte Carlo method.

9. Analytical propagation

We deal with analytical approach to uncertainty propagation only in extremely rare cases. This approach is based on the theorem that distribution of the sum of two random variables is a convolution of these distributions (if f and g are probability density function (PDF) of independent random variables X and Y, then $f * g$ is the PDF of the random variable $X + Y$). The formula for the convolution of functions f and g has the form:

$$h(x) = f * g = \int_{-\infty}^{\infty} f(x-t)g(t)dt \tag{26}$$

Practically the only examples of application of measurement uncertainty in the analysis are:

- determination of the distribution of the sum (also the difference) of any number of random variables with normal distributions; the sum (the difference) has a normal distribution of the expected value equal to the sum (the difference) of the expected values and a standard deviation equal to the geometric sum of the standard deviations,
- determination of the distribution of the sum of two random variables with uniform distributions; the sum of two random variables with uniform distributions has a trapezoidal distribution (see *e.g.* [17, ch. S10]), and in a specific case (the sum of two random variables with the same uniform distributions) has a triangular distribution.

10. GUM uncertainty framework. Law of propagation of uncertainty

This is the case if the conditions of the central limit theorem are met: a sum of a large number of random variables (regardless of their distributions) has a normal distribution with expectation equal to the sum of the expectations, and a standard deviation equal to the geometric sum of the standard deviations. The requirement of a ”large” number of elements may be alleviated to the requirement that at least 2 or 3 largest elements had standard deviations of a similar order of magnitude.

It needs to be noted explicitly that this is about the sum of random variables. In case of linear models the conditions of the central limit theorem need to be referred to products of function coefficients (sensitivity coefficients) and standard deviations. In case of nonlinear functions their earlier linearization is necessary, and the conditions of the central limit theorem remain the same as for a linear function. Expansion of the function into a Taylor series is most frequently used for linearization, which is frequently not even mentioned – values of the partial derivatives of the measurement function are simply assumed as sensitivity coefficients:

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \cdot u_c(x_i) \right)^2 \tag{27}$$

Partial derivatives present in the formula are called sensitivity coefficients and are marked as c_i. The described method of conduct (multiplication of input quantities uncertainties by relevant sensitivity coefficients and calculation of the output quantity uncertainty as a geometric sum of these products) is GUF. JCGM 101 [14] defines ”GUF” as ”*application of the law of propagation of uncertainty and the characterization of the output quantity by a Gaussian distribution or a scaled and shifted t-distribution in order to provide a coverage interval”*.

The term ”*law of propagation of uncertainty*” is described in GUM [13, ch. 5.1, 5.2]. The simplest form of the formula for the propagation of uncertainty is the formula (34) [13, formula (10)]. It applies in case of uncorrelated input quantities. In case of significant non-linearity of the measurement function, a second component should be added in the form: [13, ch. 5.1.2]:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left[\frac{1}{2} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + \frac{\partial f}{\partial x_i} \frac{\partial^3 f}{\partial x_i \partial x_j^2} \right] u^2(x_i) u^2(x_j) \tag{28}$$

For the case of correlated input quantities the formula for the propagation of uncertainty has the form [13, formula (16)]:

$$u_c^2(y) = \sum_{i=1}^{N} c_i^2\, u^2(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_i c_j u(x_i) u(x_j) r(x_i, x_j) \tag{29}$$

From this formula it follows that in case of negative correlation the value of the uncertainty calculated according to the formula can be lower than that calculated according to the formula (34). In an extreme case, when all input quantities are correlated and correlation coefficients are equal to +1, the above formula is simplified to the following form (instead of the geometric sum present in the formula (34) there is an algebraic sum):

$$u_c^2(y) = \left(\sum_{i=1}^{N} c_i u(x_i)\right)^2 = \left(\sum_{i=1}^{N} \frac{\partial f}{\partial x_i} u(x_i)\right)^2 \tag{30}$$

In ISO 14253-2 there is a following formula for uncertainty [17 , combined formulae (18) and (19)]:

$$u_c = \sqrt{\left(\sum_{i=1}^{r} u_i\right)^2 + \sum_{i=1}^{p} u_i^2} \tag{31}$$

which means that some component uncertainties are summed arithmetically (there are r of them and they are correlated components), and others (uncorrelated, there are p of them) geometrically. This is due to the fact that the standard suggests (for reasons of simplification) using only three values of the correlation coefficient: 0, +1 or -1 [18, ch. 5], although lack of correlation was assumed in all the examples provided in this standard.

According to ISO 14253-2 [18] for transparent box models measurement uncertainty is calculated according to the formula:

$$u_c = \sqrt{\left(\sum_{i=1}^{r} \frac{\partial Y}{\partial X_i} u_{Xi}\right)^2 + \sum_{i=1}^{p} \left(\frac{\partial Y}{\partial X_i} u_{Xi}\right)^2} \tag{32}$$

In comparison with the formula relating to the "black box" partial derivatives were added, which were previously equal to one.

11. Summarising

The main objective of the stage called summarising is calculation of expanded uncertainty in order to obtain the possibility to record the measurement result in the form of $y \pm U$. If the measurement result will be used as input quantity for another measuring task, knowledge of standard uncertainty u or expansion coefficient k (in calibration certificates these are U and k) is also needed. The expansion probability p, most frequently amounting to 0.95 is important additional information.

The most frequently applied approach to propagation and summarising is GUF or, in other words, the law of uncertainty propagation. Then the whole analysis is most often presented in the form of an uncertainty budget. The uncertainty budget contains names and values of all uncertainty components as well as other information, such as sensitivity coefficients or the name of the applied method of particular components determination (type A or type B evaluation). As a side note: a clear distinction should be made between the Monte Carlo method used for distributions propagation and simulation (maybe also with the Monte Carlo method) used to determine the probability distribution of the uncertainty component (determination of distribution of roundness deviation of the measured workpiece described in [26] is a good example). Examples of uncertainty budgets can be found in, *e.g.* [17, 18], there are no examples in GUM [13].

In case of using the Monte Carlo method the direct result of propagation is constituted by empirical distribution of the output quantity Y, from which all necessary information can be obtained. The whole document JCGM 101 [14] is devoted to the issue of uncertainty propagation with the Monte Carlo method. However, it should be noted that it lacks information concerning the methods of identification of probability distribution and estimation of their parameters. Without going into further details: the type of probability distribution is most frequently recognised visually on the basis of a histogram, and distribution parameters are calculated with the method of moments or the method of the maximum likelihood estimation. However, a considerable number of observations is necessary to plot a histogram. Another possibility is to use probability nets available for normal distribution, but also for other distributions, *e.g.* for the Weibull distribution. Appropriate tools are available in software containing statistical functions (Minitab, Python). The term ”maximum likelihood” related to one of the methods of probability distributions parameters estimation occurs in [35]. An example of development of the results of analytical propagation of distributions can be found in [17, ch. S10].

The above statements relate to the most frequently occurring cases, when the coverage interval is symmetrical. There are numerous situations for which the coverage interval is not symmetrical to the measurement result, however, in standards lack appropriate examples.

From the perspective of uncertainty propagation, the Monte Carlo method is a universal approach. In cases where we encounter asymmetrical distributions, the Monte Carlo method becomes the only feasible solution. The publication [14], which describes random number generators for various probability distributions, enables the resolution of more complex tasks related to measurement uncertainty determination.

12. Conclusions

In the machinery industry, it is required to specify and document the method of determining the uncertainty of all measurements that affect the decision on whether a product is compliant with the requirements or should be rejected as non-compliant. To achieve this, it is necessary to systematize and standardize documents of the status of international standards. The GUM guide, along with its supplements, largely organizes the subject of determining measurement uncertainty. From the perspective of industry professionals, documents that facilitate the creation of procedures for determining uncertainty for a wide range of measurement tasks and equipment are needed.

In coordinate measurements, a troublesome issue is the correlation between the input quantities, which are the coordinates of probed points resulting from the sampling of a large cloud of points. This problem can be solved by building measurement models based on the minimum mathematically required number of points. Such models also allow for the consideration of the specifics of individual geometric characteristics of the measured objects. Analyses performed on the developed models indicate a considerable range of measurement uncertainty for different characteristics measured by the same CMM. In the briefly described new method of estimating the uncertainty of coordinate measurements, conclusions from the analysis were used.

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Mirosław Wojtyła obtained his Ph.D. in 2010. He is currently an assistant professor at the Laboratory of Metrology at the Faculty of Mechanical Engineering and Computer Science of the University of Bielsko-Biała. His research activity focuses on coordinate measurements and estimation of uncertainty of coordinate measurements, in particular by the sensitivity analysis method.

Wojciech Płowucha received the Ph.D. degree from the University of Bielsko-Biała, Poland, in 2006. He is currently an Associate Professor at the Laboratory of Metrology of the University of Bielsko-Biała. His current research interests include GPS, measurement uncertainty of coordinate measurements. He took part in 6 EU-founded projects (including 3 as coordinator) concerning metrology education. In 2018-2022, he participated in EMPIR-founded research project no. 17NRM03 "EUCoM" as the leader of the UBB team.